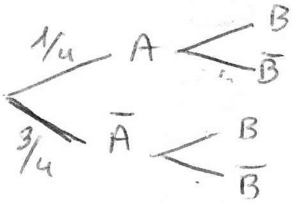
Correction DM 7

Exercice 1 : On note A : "On choisit le jeu de 32 cartes"
B : "On tire la Dame de Coeur".



- Formule des probabilités totales

$$P(B) = P(A\cap B) + P(\bar{A}\cap B)$$
$$= P(A)\,P_A(B) + P(\bar{A})\,P_{\bar{A}}(B)$$
$$= \frac{1}{4}\times\frac{1}{32} + \frac{3}{4}\times\frac{1}{52}$$
$$P(B) = \frac{1}{4}\left(\frac{1}{32}+\frac{3}{52}\right)$$
$$\boxed{P(B) = \frac{1}{4}\left(\frac{13}{416}+\frac{24}{416}\right) = \frac{37}{1664}}$$

On cherche $P_B(A) = \frac{P(A)}{P(B)}\,P_A(B)$

$$P_B(A) = \frac{1/4}{\frac{37}{1664}}\times\frac{1}{32}$$
$$P_B(A) = \frac{1}{4}\times\frac{1}{32}\times\frac{1664}{37}$$
$$\boxed{P_B(A) = \frac{13}{37}}$$

Exercice 2

① · $\Omega = [\![1;6]\!]^3$ donc $\text{card}(\Omega) = 6^3$

A : "On obtient 2 fois le numéro 4".

Pour calculer card A, il y a 3 cas :

$\underbrace{(4,4,x)}_{5 \text{ possibilités}}$ ou $\underbrace{(4,x,4)}_{5 \text{ poss}}$ ou $\underbrace{(x,4,4)}_{5 \text{ poss}}$ avec x qui ne vaut pas 4.

On a alors $\text{card}(A) = 3\times 5 = 15$.

$$\boxed{P(A) = \frac{\text{card}(A)}{\text{card}(\Omega)} = \frac{15}{6^3} = \frac{5}{72}}$$

On note A_i : "obtenir un 4 au i^{eme} lancer"

$$A = (A_1 \cap A_2 \cap \overline{A_3}) \cup (A_1 \cap \overline{A_2} \cap A_3) \cup (\overline{A_1} \cap A_2 \cap A_3)$$

$$P(A) = 3 \times P(A_1) \times P(A_2) P(\overline{A_3})$$

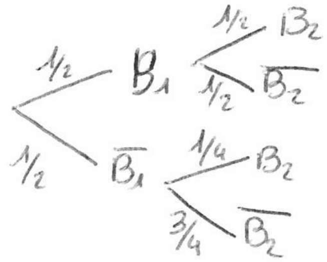
$$= 3 \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$$

$$\boxed{P(A) = \frac{5}{72}}$$

Exercice 3

(1) $P_1 = p(B_1)$: "probabilité d'avoir une boule blanche au 1^{er} tirage".

$$\boxed{P_1 = \frac{1}{2}}$$



$(B_1, \overline{B_1})$ est un système complet d'évènements

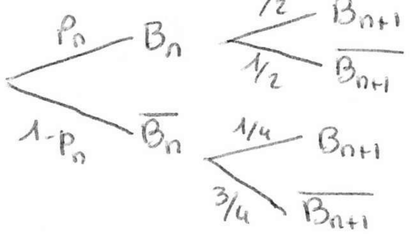
$$p(B_2) = p(B_1) p_{B_1}(B_2) + p(\overline{B_1}) p_{\overline{B_1}}(B_2)$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{8}$$

$$\boxed{p(B_2) = \frac{3}{8}}$$

(3)



$(B_n, \overline{B_n})$ est un système complet d'évènement

$$P(B_{n+1}) = p(B_n) \times p_{B_n}(B_{n+1}) + p(\overline{B_n}) p_{\overline{B_n}}(B_{n+1})$$

$$= P_n \times \frac{1}{2} + (1 - P_n) \frac{1}{4}$$

$$= \frac{P_n}{2} - \frac{P_n}{4} + \frac{1}{4}$$

$$\boxed{P_{n+1} = \frac{1}{4} P_n + \frac{1}{4}}$$

(2) $P(B_1) \times P(B_2) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

et $P(B_1 \cap B_2) = \frac{1}{4}$ donc $p(B_1) \times p(B_2) \neq p(B_1 \cap B_2)$

et donc B_1 et B_2 ne sont pas indépendant

④ On résout $x = \frac{1}{4}x + \frac{1}{4}$

$\Leftrightarrow \frac{3}{4}x = \frac{1}{4}$

$\Leftrightarrow \boxed{x = \frac{1}{3}}$.

On pose alors $v_n = p_n - \frac{1}{3}$ et $\forall n \in \mathbb{N}$,

$v_{n+1} = p_{n+1} - \frac{1}{3}$.

$\Leftrightarrow v_{n+1} = \frac{1}{4}p_n + \frac{1}{4} - \frac{1}{3}$.

$\Leftrightarrow v_{n+1} = \frac{1}{4}\left(v_n + \frac{1}{3}\right) - \frac{1}{12}$.

$\Leftrightarrow v_{n+1} = \frac{1}{4}v_n$ Donc $(v_n)_n$ est une suite géométrique

$v_n = v_1\left(\frac{1}{4}\right)^{n-1}$ et $v_1 = p_1 - \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

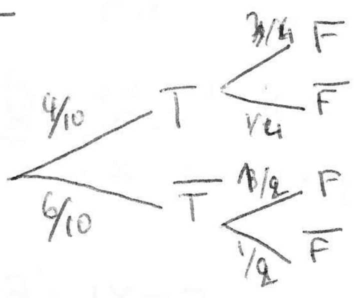
$v_n = \frac{1}{6} \times \left(\frac{1}{4}\right)^{n-1} \Leftrightarrow \boxed{p_n = \frac{1}{6} \times \left(\frac{1}{4}\right)^{n-1} + \frac{1}{3}}$

Comme $\frac{1}{4} < 1$, $\lim_{n \to +\infty} \left(\frac{1}{4}\right)^{n-1} = 0$

et $\boxed{\lim_{n \to +\infty} p_n = \frac{1}{3}}$

Exercice 4

① (a)



$(T, \overline{T})$ est un système complet d'événements

$p(F) = p(T) \times p_T(F) + p(\overline{T})p_{\overline{T}}(F)$

$= \frac{4}{10} \times \frac{3}{4} + \frac{6}{10} \times \frac{1}{2}$.

$\boxed{p(F) = \frac{3}{10} + \frac{3}{10} = \frac{6}{10}}$

(b) $p_F(T) = \frac{p(T)}{p(F)} p_T(F) = \frac{4/10}{6/10} \times \frac{3}{4} = \frac{4}{10} \times \frac{3}{4} \times \frac{10}{6} \boxed{= \frac{1}{2}}$.

② (a) $p_{\overline{T}}(E) = \left(\frac{1}{2}\right)^4$ (loi binomiale).

(b) $p_T(\overline{E}) = 1 - \left(\frac{3}{4}\right)^4$

(c) On cherche $p((\overline{T} \cap E) \cup (T \cap \overline{E}))$

$$p((\overline{T} \cap E) \cup (T \cap \overline{E})) = p(\overline{T} \cap E) + p(T \cap \overline{E})$$

$$= p(\overline{T})\, p_{\overline{T}}(E) + p(T) \times p_T(\overline{E}).$$

$$= \frac{6}{10} \times \left(\frac{1}{2}\right)^4 + \frac{4}{10} \times \left(1 - \left(\frac{3}{4}\right)^4\right).$$

$$= \frac{3}{80} + \frac{4}{10} - \frac{81}{640}$$

$$= \frac{35}{80} - \frac{81}{640} = \frac{199}{640}$$

(d) $p_T(E) = \left(\frac{3}{4}\right)^4$

$$p(E) = p(\overline{T}) \times p_{\overline{T}}(E) + p(T) \times p_T(E)$$

$$= \frac{6}{10} \times \left(\frac{1}{2}\right)^4 + \frac{4}{10} \times \left(\frac{3}{4}\right)^4$$

$$= \frac{3}{80} + \frac{81}{640}$$

$$\boxed{p(E) = \frac{105}{640} = \frac{21}{108}}.$$